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Killing spinor equations in dimension 7 and geometry of integrable G_2-manifolds

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Abstract

We compute the scalar curvature of seven-dimensional G_2-manifolds admitting a G_2-connection with totally skew-symmetric torsion. We prove the formula for the general solution of the Killing spinor equation and express the Riemannian scalar curvature of the solution in terms of the dilation function and the NS 3-form field. In dimension $n = 7$ the dilation function involved in the second fermionic string equation has an interpretation as a conformal change of the underlying integrable G_2-structure into a cocalibrated one of pure type W_3.

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1. Introduction

Riemannian manifolds admitting parallel spinors with respect to a metric connection with totally skew-symmetric torsion became a subject of interest in theoretical and mathematical physics recently. One of the main reasons is that the number of preserved supersymmetries in string theory depends essentially on the number of parallel spinors. In 10-dimensional string theory, the Killing spinor equations with non-constant dilation Φ and the 3-form field strength H can be written in the following way [37] (see [16,22,23]):

$$\nabla \Psi = 0, \qquad (d\Phi - \tfrac{1}{2}H) \cdot \Psi = 0, \tag{$*$}$$

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where Ψ is a spinor field and ∇ a metric connection with totally skew-symmetric torsion $T = H$. The existence of a parallel spinor imposes restrictions on the holonomy group since the spinor holonomy representation has to have a fixed point. In the case of the torsion-free metric connection (the Levi–Civita connection),the possible Riemannian holonomy groups are known to be $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, G_2, $\mathrm{Spin}(7)$ [28,39]. The Riemannian holonomy condition imposes strong restrictions on the geometry and leads to the consideration of Calabi–Yau manifolds, hyper-Kähler manifolds, parallel G_2-manifolds and parallel $\mathrm{Spin}(7)$-manifolds. All of them are of great interest in mathematics (see [26] for detailed discussions) as well as in high-energy physics and string theory [31]. However, it seems that the geometry of these spaces is too restrictive for various problems in string theory [20,30,35]. One possible generalization of Calabi–Yau manifolds, hyper-Kähler manifolds, parallel G_2-manifolds and parallel $\mathrm{Spin}(7)$-manifolds are manifolds equipped with linear metric connections having skew-symmetric torsion and holonomy contained in $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, G_2, $\mathrm{Spin}(7)$. One remarkable fact is that the existence (in small dimensions) of a parallel spinor with respect to a metric connection ∇ with skew-symmetric torsion determines the connection in a unique way if its holonomy group is a subgroup of SU, Sp, G_2, provided that some additional differential conditions on the structure are fulfilled [16,37], and always in dimension 8 for a subgroup of the group $\mathrm{Spin}(7)$ [21]. The case of 16-dimensional Riemannian manifolds with $\mathrm{Spin}(9)$-structure was investigated in [13], homogeneous models are discussed in [2]. The existence of ∇-parallel spinors in dimensions 4–8 is studied in [10,16,17,21,22,37]. In dimension 7, the first consequence is that the manifold should be a G_2-manifold with an integrable G_2-structure [16], i.e., the structure group could be reduced to the group G_2 and the corresponding 3-form ω^3 should obey $d * \omega^3 = \theta \wedge *\omega^3$ for some special 1-form θ. In this paper we study solutions to the Killing spinor equations $(*)$ in dimension 7 and the geometry of integrable G_2-manifolds. We find a formula for the Riemannian scalar curvature in terms of the fundamental 3-form. Our first main result is the following theorem.

Theorem 1.1. *Let (M, g, ω^3) be an integrable G_2-manifold with the fundamental 3-form ω^3. The Riemannian scalar curvature Scal^g is given in terms of the fundamental 3-form ω^3 by*

$$\mathrm{Scal}^g = \frac{1}{18}(d\omega^3, *\omega^3)^2 + 2\|\theta\|^2 - \frac{1}{12}\|T\|^2 + 3\delta\theta, \tag{1.1}$$

where θ and T are the Lee form and the torsion of the unique G_2-connection given by

$$T = - * d\omega^3 + \frac{1}{6}(d\omega^3, *\omega^3)\cdot\omega^3 + *(\theta\wedge\omega^3),$$
$$\theta = -\frac{1}{3} * (*d\omega^3 \wedge \omega^3) = \frac{1}{3} * (\delta\omega^3 \wedge *\omega^3). \tag{1.2}$$

We remark that the torsion form T was been computed in [16]. Returning to the Killing spinor equations $(*)$, we present necessary and sufficient conditions for a G_2-manifold to be a solution to both of them. In fact we show that the dilation function arises from the Lee 1-form. Finally, we give a formula for the Riemannian scalar curvature of any solution to both Killing spinor equations in dimension 7. Our second main result is the following theorem.

Theorem 1.2. *In dimension 7 the following conditions are equivalent*:

(1) *The Killing spinor equations $(*)$ admit a solution with dilation Φ.*

(2) *There exists an integrable G_2-structure (g, ω^3) with closed Lee form, which is locally conformally equivalent to a cocalibrated G_2-structure of pure type W_3.*

More precisely, the structure is determined by the equations:

$$d * \omega^3 = \theta \wedge *\omega^3, \quad (d\omega^3, *\omega^3) = 0, \quad \theta = -2d\Phi \tag{1.3}$$

and the NS 3*-form* $H = T$ *is given by*

$$T = - * d\omega^3 - 2 * (d\Phi \wedge \omega^3). \tag{1.4}$$

The Riemannian scalar curvature is determined by

$$\mathrm{Scal}^g = 8 \cdot \|d\Phi\|^2 - \tfrac{1}{12} \cdot \|T\|^2 - 6 \cdot \triangle\Phi, \tag{1.5}$$

where $\triangle\Phi = \delta d\Phi$ *is the Laplacian. The solution has constant dilation if and only if the* G_2*-structure is cocalibrated of pure type* W_3.

Our proof relies on the existence theorem for a G_2-connection with torsion, the Schrödinger–Lichnerowicz formula for the connection with torsion (both established in [16]) and the special properties of the Clifford action on the special parallel spinor.

2. General properties of G_2-structures

Let us consider $\mathbb{R}^7$ endowed with an orientation and its standard inner product. Denote an oriented orthonormal basis by $e_1, \dots, e_7$. We shall use the same notation for the dual basis. We denote the monomial $e_i \wedge e_j \wedge e_k$ by e_{ijk}. Consider the 3-form ω^3 on $\mathbb{R}^7$ given by

$$\omega^3 = e_{127} + e_{135} - e_{146} - e_{236} - e_{245} + e_{347} + e_{567}. \tag{2.1}$$

The subgroup of SO(7) that fixes ω^3 is the exceptional Lie group G_2. It is a compact, simply connected, simple Lie group of dimension 14 [32]. The 3-form ω^3 corresponds to a real spinor and, therefore, G_2 is the isotropy group of a non-trivial real spinor. A G_2*-structure* on a 7-manifold M^7 is a reduction of the structure group of the tangent bundle to the exceptional group G_2. This can be described geometrically by a nowhere vanishing differential 3-form ω^3 on M^7, which can be locally written as (2.1). The 3-form ω^3 is called the *fundamental form* of the G_2-manifold M^7 (see [3]) and it determines the metric completely. The action of G_2 on the tangent space gives an action of G_2 on k-forms and we obtain the following splitting [6,11]:

$$\Lambda^1(M^7) = \Lambda^1_7, \qquad \Lambda^2(M^7) = \Lambda^2_7 \oplus \Lambda^2_{14}, \qquad \Lambda^3(M^7) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27},$$

where

$$\Lambda^2_7 = \{\alpha \in \Lambda^2(M^7) | * (\alpha \wedge \omega^3) = 2\alpha\}, \quad \Lambda^2_{14} = \{\alpha \in \Lambda^2(M^7) \| * (\alpha \wedge \omega^3) = -\alpha\},$$
$$\Lambda^3_7 = \{*(\beta \wedge \omega^3) | \beta \in \Lambda^1(M^7)\}, \quad \Lambda^3_{27} = \{\gamma \in \Lambda^3(M^7) | \gamma \wedge \omega^3 = 0, \ \gamma \wedge *\omega^3 = 0\}.$$

Following [8] we consider the 1-form θ defined by

$$3\theta = - * (*d\omega^3 \wedge \omega^3) = *(\delta\omega^3 \wedge *\omega^3). \tag{2.2}$$

We shall call this 1-form the *Lee form* associated with a given G_2-structure. If the Lee form vanishes, then we shall call the G_2-structure *balanced*. The classification of the different types of G_2-structures was worked out by Fernandez–Gray [11], and Cabrera used the Lee form to characterize each of the 16 classes. An *integrable* G_2-structure (or a structure of type $W_1 \oplus W_3 \oplus W_4$) is characterized by the differential equation

$$d * \omega^3 = \theta \wedge *\omega^3$$

and a *cocalibrated* G_2-structure is defined by the condition

$$d * \omega^3 = 0.$$

A *cocalibrated* G_2*-structure of pure type* W_3 is characterized by the two conditions $d*\omega^3 = 0$, $d\omega^3 \wedge \omega^3 = 0$. Then the following proposition follows immediately.

Proposition 2.1. *If the Lee* 1*-form is closed, then the* G_2*-structure is locally conformal to a balanced* G_2*-structure.*

We shall call locally conformally parallel G_2-manifolds that are not globally conformally parallel *strict locally conformally parallel*.

Example 2.1. Any seven-dimensional oriented spin Riemannian manifold admits a certain G_2-structure, in general a non-parallel one (see for example [27]). The first known examples of complete parallel G_2-manifold were constructed by Bryant and Salamon [7], the first compact examples by Joyce [24–26]. There are many known examples of compact nearly parallel G_2-manifolds: S^7 [11], $\mathrm{SO}(5)/\mathrm{SO}(3)$ [7,33], the Aloff–Wallach spaces $N(g,l) = \mathrm{SU}(3)/U(1)_{g,l}$ [9], any Einstein–Sasakian and any 3-Sasakian space in dimension 7 [14,15]. There are also some non-regular 3-Sasakian manifolds (see [4,5]). Moreover, compact nearly parallel G_2-manifolds with large symmetry group are classified in [15]. Compact integrable nilmanifolds are constructed and studied in [12]. Any minimal hypersurface N in $\mathbb{R}^8$ admits a cocalibrated G_2-structure [11]. Moreover, the structure is parallel, nearly parallel, cocalibrated of pure type if and only if the hypersurface N is totally geodesic, totally umbilic or minimal, respectively.

3. Conformal transformations of G_2-structures

We study the conformal transformation of G_2-structures (see [11]).

Proposition 3.1. *Let* $\bar{g} = \mathrm{e}^{2f} \cdot g$, $\bar{\omega}^3 = \mathrm{e}^{3f} \cdot \omega^3$ *be a conformal change of a* G_2*-structure* (g, ω^3) *and denote by* $\bar{\theta}, \theta$ *the corresponding Lee forms, respectively. Then*

$$\bar{\theta} = \theta + 4df. \tag{3.1}$$

Proof. We have the relations

$$\mathrm{vol}_{\bar{g}} = \mathrm{e}^{7f} \cdot \mathrm{vol}_g, \qquad d\bar{\omega}^3 = \mathrm{e}^{3f} \cdot (3df \wedge \omega^3 + d\omega^3).$$

We calculate

$$\bar{*}d\bar{\omega}^3 = \mathrm{e}^{4f}(*d\omega^3 + 3*(df\wedge\omega^3)),\quad \bar{*}d\bar{\omega}^3\wedge\bar{\omega}^3 = \mathrm{e}^{7f}(*d\omega^3\wedge\omega^3 - 12*df),$$

where we used the general identity $*(\omega^3\wedge\gamma)\wedge\omega^3 = 4*\gamma$, which is valid for any 1-form γ. Consequently, we obtain $\bar{\theta} = -(1/3)\bar{*}(\bar{*}d\bar{\omega}^3\wedge\bar{\omega}^3) = -(1/3)(*(*d\omega^3\wedge\omega^3) - 12*^2 df) = \theta + 4df$. □

Proposition 3.1 allows us to find a distinguished G_2-structure on a compact seven-dimensional G_2-manifold.

Theorem 3.1. *Let* (M^7, g, ω^3) *be a compact* 7*-dimensional* G_2*-manifold. Then there exists a unique* (*up to homothety*) *conformal* G_2*-structure* $g_0 = \mathrm{e}^{2f}\cdot g$, $\omega_0^3 = \mathrm{e}^{3f}\cdot\omega^3$ *such that the corresponding Lee form is coclosed*, $\delta_0\theta_0 = 0$.

Proof. We shall use the Gauduchon theorem for the existence of a distinguished metric on a compact, Hermitian or Weyl manifold [18,19]. We shall use the expression of this theorem in terms of a Weyl structure (see [38, Appendix 1]). We consider the Weyl manifold $(M^7, g, \theta, \nabla^W)$ with the Weyl 1-form θ, where ∇^W is a torsion-free linear connection on M^7 determined by the condition $\nabla^W g = \theta\otimes g$. Applying the Gauduchon theorem we can find, in a unique way, a conformal metric g_0 such that the corresponding Weyl 1-form is coclosed with respect to g_0. The key point is that, by Proposition 3.1, the Lee form transforms under conformal rescaling according to (3.1), which is exactly the transformation of the Weyl 1-form under conformal rescaling of the metric $\bar{g} = \mathrm{e}^{4f}\cdot g$. Thus, there exists (up to homothety) a unique conformal G_2-structure (g_0, ω_0^3) with coclosed Lee form. □

We shall call the G_2-structure with coclosed Lee form *the Gauduchon* G_2*-structure*.

Corollary 3.1. *Let* (M^7, g, Φ) *be a compact* G_2*-manifold and* (g, Φ) *be the Gauduchon structure. Then the following formula holds*:

$$*(d\delta\omega^3\wedge *\omega^3) = \|\delta\omega^3\|^2.$$

In particular, *if the structure is integrable*, *then*

$$*(d\delta\omega^3\wedge *\omega^3) = 24\|\theta\|^2.$$

Proof. Using (2.2), we calculate that

$$\begin{aligned}0 = 3\cdot\delta\theta &= *d(\delta\omega^3\wedge *\omega^3) = *(d\delta\omega^3\wedge *\omega^3 - *d*\omega^3\wedge d*\omega^3)\\ &= *(d\delta\omega^3\wedge *\omega^3 - \|\delta\omega^3\|^2\cdot\mathrm{vol}).\end{aligned}$$

If the structure is integrable, then $\|\delta\omega^3\|^2 = 24\|\theta\|^2$. □

Corollary 3.2. *On a compact* G_2*-manifold with closed Lee form whose Gauduchon* G_2*-structure is not balanced, the first Betti number satisfies* $b_1(M)\geq 1$.

For integrable G_2-manifolds one can define a suitable elliptic complex as well as cohomology groups $\tilde{H}^i(M^7)$ (see [12]). The first cohomology group is given by

$$\tilde{H}^1(M^7) = \{\alpha \in \Lambda^1(M^7) : d\alpha \wedge *\omega^3 = 0, d * \alpha = 0\}.$$

Corollary 3.3. *On a compact integrable manifold which is not globally conformally balanced, one has* $\tilde{b}_1 \geq 1$.

Proof. By the condition of the theorem the Gauduchon structure has a non-identically zero Lee form. Then $0 = \delta\omega^3 = *(d\theta \wedge *\omega^3)$, since the structure is integrable. Adding the condition $\delta\theta = 0$, we obtain $\tilde{b}_1 \geq 1$. □

4. Connections with torsion, parallel spinors and Riemannian scalar curvature

The Ricci tensor of an integrable G_2-manifold was expressed in principle by the structure form ω^3 in the paper [16]. Here we intend to find an explicit formula for the Riemannian scalar curvature. Using the unique connection with skew-symmetric torsion preserving the given integrable G_2-structure found in [16], we apply the Schrödinger–Lichnerowicz formula for the Dirac operator of a metric connection with totally skew-symmetric torsion (see [16]) in order to derive the formula for the scalar curvature. First, let us summarize the mentioned results from [16].

Theorem 4.1 (see [16]). *Let* (M^7, g, ω^3) *be a* G_2*-manifold. Then the following conditions are equivalent*:

(1) *The* G_2*-structure is integrable, i.e.,* $d * \omega^3 = \theta \wedge *\omega^3$;
(2) *There exists a unique linear connection* ∇ *preserving the* G_2*-structure with totally skew-symmetric torsion* T *given by*

$$T = - * d\omega^3 + \tfrac{1}{6}(d\omega^3, *\omega^3) \cdot \omega^3 + *(\theta \wedge \omega^3). \tag{4.1}$$

Furthermore, for any integrable G_2*-structure, the projections* $\pi^4_1(d\omega^3), \pi^4_7(d\omega^3)$ *of* $d\omega^3$ *onto* Λ^4_1 *and* Λ^4_7*, respectively, are given by*

$$\pi^4_1(d\omega^3) = \tfrac{1}{7} \cdot (d\omega^3, *\omega^3) * \omega^3, \qquad \pi^4_7(d\omega^3) = \tfrac{3}{4} \cdot \theta \wedge \omega^3,$$

there exists a ∇*-parallel spinor* Ψ_0 *corresponding to the fundamental form* ω^3 *and the Clifford action of the torsion* 3*-form on it is*

$$T \cdot \Psi_0 = \tfrac{7}{6} \cdot \lambda \cdot \Psi_0 - \theta \cdot \Psi_0, \quad \lambda = -\tfrac{1}{7} \cdot (d\omega^3, *\omega^3). \tag{4.2}$$

Keeping in mind Proposition 3.1, we obtain the following corollary.

Corollary 4.1. *The Lee form of an integrable* G_2*-structure is given by* $*(\omega^3 \wedge T) = -\theta$.

Corollary 4.2. *The torsion* 3*-form T of ∇ changes by a conformal transformation* ($g_o = e^{2f} \cdot g, \omega_o^3 = e^{3f} \cdot \omega^3$) *of the G_2-structure by*

$$T_o = e^{4f} \cdot (T + *df \wedge \omega^3).$$

Let D and Scal be the Dirac operator and the scalar curvature of the G_2-connection defined as usually by

$$D = \sum_{i=1}^{7} e_i \cdot \nabla_{e_i}, \qquad \mathrm{Scal} = \sum_{i,j=1}^{7} R^{\nabla}(e_i, e_j, e_j, e_i).$$

The scalar curvature Scal^g of the metric is given by (see [16,22])

$$\mathrm{Scal}^g = \mathrm{Scal} + \tfrac{1}{4}\|T\|^2. \tag{4.3}$$

The 4-form σ^T defined by the formula

$$\sigma^T = \frac{1}{2}\sum_{i=0}^{7}(e_i \lrcorner T) \wedge (e_i \lrcorner T)$$

plays an important role in the integrability conditions for ∇-parallel spinors.

Theorem 4.2 (see [16]). *Let Ψ be a parallel spinor with respect to a metric connection ∇ with totally skew-symmetric torsion T on a Riemannian spin manifold M^n. Then the following formulas hold*

$$3 \cdot dT \cdot \Psi - 2 \cdot \sigma^T \cdot \Psi + \mathrm{Scal} \cdot \Psi = 0,$$
$$D(T \cdot \Psi) = dT \cdot \Psi + \delta T \cdot \Psi - 2 \cdot \sigma^T \cdot \Psi.$$

Proof of Theorem 1.1. Let Ψ_0 be the ∇-parallel spinor corresponding to the fundamental 3-form ω^3. Then the Riemannian Dirac operator D^g and the Levi–Civita connection ∇^g act on Ψ_0 by the rule

$$\nabla^g_X \Psi_0 = -\tfrac{1}{4}(X \lrcorner T) \cdot \Psi_0, \qquad D^g \Psi_0 = -\tfrac{3}{4} \cdot T \cdot \Psi_0 = -\tfrac{7}{8} \cdot \lambda \cdot \Psi_0 + \tfrac{3}{4} \cdot \theta \cdot \Psi_0, \tag{4.4}$$

where we used Theorem 4.1. We are going to apply the well known Schrödinger–Lichnerowicz formula [29,36]:

$$(D^g)^2 = \triangle^g + \tfrac{1}{4} \cdot \mathrm{Scal}^g, \quad \triangle^g = -\sum_{i=1}^{n}(\nabla^g_{e_i}\nabla^g_{e_i} - \nabla^g_{\nabla_{e_i} e_i})$$

to the ∇-parallel spinor field Ψ_0. The formula (4.4) yields that

$$\begin{aligned}(D^g)^2\Psi_0 &= -\tfrac{7}{8} \cdot D^g(\lambda \cdot \Psi_0) + \tfrac{3}{4} \cdot D^g(\theta \cdot \Psi_0) \\ &= (\tfrac{49}{64} \cdot \lambda^2 + \tfrac{9}{16} \cdot \|\theta\|^2 + \tfrac{3}{4} \cdot \delta\theta) \cdot \Psi_0 - \tfrac{7}{8} \cdot d\lambda \cdot \Psi_0 + \tfrac{3}{4} \cdot d\theta \cdot \Psi_0 \\ &\quad + \tfrac{3}{8} \cdot (\theta \lrcorner T) \cdot \Psi_0,\end{aligned} \tag{4.5}$$

where we used the general identity $D^g\theta+\theta D^g = d\theta+\delta\theta-2\nabla_\theta$. We compute the Laplacian $\triangle^g$. Fix a normal coordinate system at a point $p \in M^n$ such that $(\nabla_{ei}e_i)_p = 0$, use (4.4) as well as the properties of the Clifford multiplication. Then one obtains the following formula [21]:

$$\triangle^g\Psi_0 = \frac{1}{4}\cdot\sum_{i=1}^{n}\left(\nabla_{e_i}(e_i\rfloor T)\cdot\Psi_0 - \frac{1}{16}\cdot(e_i\rfloor T)\cdot(e_i\rfloor T)\cdot\Psi_0\right)$$
$$= -\frac{1}{4}\cdot\delta T\cdot\Psi_0 - \frac{1}{16}\cdot\left(2\sigma^T - \frac{1}{2}\cdot\|T\|^2\right)\cdot\Psi_0. \tag{4.6}$$

Substituting (4.5) and (4.6) into the SL-formula, multiplying the obtained result by Ψ_0 and taking the real part, we arrive at

$$\left(\tfrac{49}{64}\cdot\lambda^2 + \tfrac{9}{16}\cdot\|\theta\|^2 + \tfrac{3}{4}\cdot\delta\theta\right)\cdot\|\Psi_0\|^2$$
$$= \left(\tfrac{1}{32}{}^2 + \tfrac{1}{4}\cdot\mathrm{Scal}^g\right)\cdot\|\Psi_0\|^2 - \tfrac{1}{8}\cdot(\sigma^T\cdot\Psi_0,\Psi_0). \tag{4.7}$$

On the other hand, using (4.2), we obtain

$$D(T\cdot\Psi_0) = D\left(\frac{7}{6}\cdot\lambda\cdot\Psi_0 - \theta\cdot\Psi_0\right) = \sum_{i=1}^{n} e_i\cdot\nabla_{e_i}\left(\frac{7}{6}\cdot\lambda\cdot\Psi_0 - \theta\cdot\Psi_0\right)$$
$$= \frac{7}{6}\cdot d\lambda\cdot\Psi_0 - (d^\nabla\theta + \delta\theta)\cdot\Psi_0,$$

where d^∇ is the exterior derivative with respect to the G_2-connection ∇. Now, Theorem 4.2 gives $(7/6)d\lambda\cdot\Psi_0 - d^\nabla\theta\cdot\Psi_0 - \delta\theta\cdot\Psi_0 = dT\cdot\Psi_0 - 2\sigma^T\cdot\Psi_0 + \delta T\cdot\Psi_0$. Multiplying the latter equality by Ψ_0 and taking the real part, we obtain $-\delta\theta\cdot\|\Psi_0\|^2 = (dT\cdot\Psi_0,\Psi_0) - (2\sigma^T\cdot\Psi_0,\Psi_0)$. Consequently, Theorem 4.2 and (4.3) imply

$$\left(-3\cdot\delta\theta - \tfrac{1}{4}\cdot\|T\|^2 + \mathrm{Scal}^g\right)\cdot\|\Psi_0\|^2 + 4\cdot(\sigma^T\cdot\Psi_0,\Psi_0) = 0. \tag{4.8}$$

Finally, (4.7) and (4.8) imply (1.1) and the proof of Theorem 1.1 is complete. □

Corollary 4.3. *On a cocalibrated G_2-manifold of pure type the Riemannian scalar curvature is given by*

$$\mathrm{Scal}^g = -\tfrac{1}{12}\cdot\|d\omega^3\|^2.$$

Proof. In the case of a cocalibrated G_2-structure of pure type, the torsion 3-form $T = -*d\omega^3$. The claim follows from Theorem 1.1. □

Using the results in [11] we derive immediately the following formula, which is essentially the reformulated Gauss equation.

Corollary 4.4. *Let M^7 be a hypersurface in $\mathbb{R}^8$ with second fundamental form S and mean curvature H. Then the Riemannian scalar curvature on M^7 is given by the formula*:

$$\mathrm{Scal}^g = \tfrac{49}{18}\cdot\|H\|^2 - \tfrac{1}{12}\cdot\|S_0\|^2, \tag{4.9}$$

where S_0 is the image of the traceless part of the second fundamental form via the isomorphism $S_0^2(R^7) \to \Lambda^3_{27}$. In particular, if M is a minimal hypersurface, then

$$\mathrm{Scal}^g = -\tfrac{1}{12}\|S_0\|^2 \leq 0. \tag{4.10}$$

Theorem 4.3. *Let M^7 be a compact, connected spin 7-manifold with a fixed orientation. If it admits a strictly locally conformally parallel G_2-structure, then*:

(1) *M admits a Riemannian metric g_Y with strictly positive constant scalar curvature*.
(2) *The first Betti number is at least 1*, $b_1(M) \geq 1$.

Proof. We have $\|T\|^2 = (3/2)\|\theta\|^2$ since the structure is locally conformally parallel. Then, Theorem 1.1 leads to the formula:

$$\mathrm{Scal}^g = \tfrac{15}{8} \cdot \|\theta\|^2 + 3 \cdot \delta\theta. \tag{4.11}$$

According to the solution of the Yamabe conjecture [34] there is a metric $g_Y = \mathrm{e}^{2f} \cdot g$ in the conformal class of g with constant scalar curvature. Consider the locally conformally parallel G_2-structure $(g_Y = \mathrm{e}^{2f} \cdot g, \omega^3_Y = \mathrm{e}^{3f} \cdot \omega^3)$. The equality (4.11) also holds for the structure (g_Y, ω^3_Y) and an integration over M gives

$$\mathrm{Scal}^{g_Y} \cdot \mathrm{vol}(g_Y) = \frac{11}{6} \int_M \|\theta\|^2 d\,\mathrm{vol} > 0,$$

since the structure is strictly locally conformally parallel. The second assertion is a consequence of Corollary 3.2. □

5. Solutions to the Killing spinor equations in dimension 7

We consider the Killing spinor equations (∗) in dimension 7. The existence of a ∇-parallel spinor is equivalent to the existence of a ∇-parallel integrable G_2-structure and the 3-form field strength $H = T$ is given by (4.1). We now investigate the second Killing spinor equation (∗).

Proof of Theorem 1.2. Let Ψ be an arbitrary ∇-parallel such that $(d\Phi - T) \cdot \Psi = 0$. The spinor field Ψ defines a second G_2-structure ω^3_0 such that $\Psi = \Psi_0$ is the canonical spinor field. Since the connection preserves the spinor field Ψ, it preserves the G_2-structure ω^3_0, too. On the other hand, the connection preserving ω^3_0 is unique. Consequently, the torsion T_0 coincides with the torsion form T and for the G_2-structure ω^3_0 we have

$$\nabla\Psi_0 = 0, \qquad (d\Phi - \tfrac{1}{2}T_0) \cdot \Psi_0 = 0.$$

The Clifford action $T_0 \cdot \Psi_0$ depends only on the $(\Lambda^3_1 \oplus \Lambda^3_7)$-part of T_0. Using (4.1) and the algebraic formulas

$$*(\gamma \wedge \omega^3_0) \cdot \Psi_0 = -\gamma \lrcorner (*\omega^3_0) \cdot \Psi_0 = -4 \cdot \gamma \cdot \Psi_0, \qquad \omega^3_0 \cdot \Psi_0 = -7 \cdot \Psi_0$$

we calculate

$$T_0 \cdot \Psi_0 = -\theta \cdot \Psi_0 - \tfrac{1}{6} \cdot (d\omega_0^3, *\omega_0^3) \cdot \Psi_0. \tag{5.1}$$

Comparing with the second Killing spinor equation $(*)$ we find $2 \cdot d\Phi = -\beta$, $(d\omega_0^3, *\omega_0^3) = 0$ which completes the proof. □

As a corollary we obtain the result from [20], which states that any solution to both equations $(*)$ has necessarily the NS three form $H = T$ given by (1.4). A more precise analysis using Proposition 3.1 and Theorem 1.1 of the explicit solutions constructed in [20] shows that these solutions are conformally equivalent to a cocalibrated structure of pure type. In other words, the multiplication of the G_2-structures $(g^{\pm}, \omega^{3\pm})$ by $(\mathrm{e}^{\Phi} \cdot g^{\pm}, \mathrm{e}^{(3/2)\Phi} \cdot \omega^{3\pm})$ is a new example of a cocalibrated G_2-structure of pure type W_3, and it is a solution to the Killing spinor equations with constant dilation. The same conclusions are valid for the solutions constructed in [1,30,35].

Theorem 1.2 allows us to construct a lot of compact solutions to the Killing spinor equations. If the dilation is a globally defined function, then any solution is globally conformally equivalent to a cocalibrated G_2-structure of pure type. For example, any conformal transformation of a compact seven-dimensional manifold with a Riemannian holonomy group G_2 constructed by Joyce [24,25] is a solution with non-constant dilation. Another source of solutions are conformal transformations of the cocalibrated G_2-structures of pure type W_3 induced on any minimal hypersurface in $\mathbb{R}^8$. Summarizing, we obtain:

Corollary 5.1. *Any solution (M^7, g, ω^3) to the Killing spinor equations $(*)$ in dimension 7 with non-constant globally defined dilation function Φ comes from a solution with constant dilation by a conformal transformation $(g = \mathrm{e}^{\Phi} \cdot g_0, \omega^3 = \mathrm{e}^{(3/2)\Phi} \cdot \omega_0^3)$, where (g_0, ω_0^3) is a cocalibrated G_2-structure of pure type W_3.*

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